Optimal conclusive teleportation of a d-dimensional unknown state

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(February 1, 2008)

We formulate a conclusive teleportation protocol for a system in d-dimensional Hilbert space utilizing the positive operator valued measurement at the sending station. The conclusive teleportation protocol ensures some perfect teleportation events when the channel is only partially entangled, at the expense of lowering the overall average fidelity. We find the change of the fidelity as optimizing the conclusive teleportation events and discuss how much information remains in the inconclusive parts of the teleportation.

PACS number(s); 03.67.-a, 89.70.+c

I. INTRODUCTION

Quantum teleportation [1,2] serves probably the best test ground for quantum entanglement. When its quantum channel is optimally entangled, quantum teleportation faithfully transmits the quantum state of a system. It has been found for a spin-1/2 system that even when the quantum channel is not maximally entangled, quantum teleportation provides better transmission of a quantum state than any classical communication protocol [3].

When an imperfect quantum channel is given, there are two components we can alter in quantum teleportation to satisfy the needs: *measurement* at the sending station and *transformation* at the receiving station. Banaszek found that, when the channel is *pure*, a joint von-Neumann measurement and a unitary transformation maximizes the average fidelity for the teleportation of a state in d-dimensional Hilbert space [4]. In some cases, this may not be the optimal strategy for particular purposes. For example, when a perfect replica of the original state is only of use, maximizing the average fidelity does not have to be the best strategy because it can mean none of the replica states are perfect even though the density matrix of the replica ensemble maximally resembles the original state. In this paper we are interested in how to optimize the successful teleportation of a quantum state in d-dimensional Hilbert space.

Recently, Mor and Horodecki [5] proposed a conclusive teleportation protocol for a quantum state in two-dimensional Hilbert space by employing a positive operator valued measurement (POVM) [6] as the joint measurement at the sending station. When the measurement is successful by a random chance, the initially unknown quantum state is teleported perfectly, which is called a *conclusive* event. In contrast, for an *inconclusive* event, the sender can extract no information from the measurement outcome and the teleportation loses its quantum characteristics. To indicate the success of the measurement, an extra single-bit has to be sent to the receiver together with the two-bit information via a classical channel. Bandyopadhyay [7] and Li *et al.* [8] have also proposed protocols to implement conclusive teleportation in two-dimensional Hilbert space. Bandyopadhyay uses a combination of orthogonal GHZ measurements and POVM's while Li *et al.* take general transformations at the receiving station leaving the measurement orthogonal.

Quantum information theory has been extensively developed for a two-level spin-1/2 system because this simple model provides a natural extension of a binomial bit to a quantum bit, namely, *qubit*, and gives a comprehensive understanding of fundamental quantum theory. It is only recently that d-dimensional quantum systems have attracted a considerable research effort. Quantum cloning in d dimensions was studied by Zanardi [9] and quantum teleportation by Zubairy [10] and Stenholm and Bardroff [11]. Rungta *et al.* called the d-dimensional quantum system the *qudit* and investigated its entanglement and separability [12]. The qudit was extensively studied as a finite-dimensional version of a continuous variable state by Gottesman *et al.* [13].

The POVM is not a new concept in quantum information theory. The POVM is one of the standard procedures for entanglement purification and concentration. The entanglement purification is a protocol to select the subensemble of maximally entangled pairs from the whole ensemble of quantum states [14]. In the purification protocol, two distant observers employ only local measurements and classical communication together with the post-selection of subensembles. The local measurements may involve POVM's [15]. The teleportation procedure may follow the purification of the quantum channel for the faithful transmission of a quantum system. The optimal POVM has recently been implemented in a laboratory for a two-level system [16].

In this paper, we formulate the conclusive teleportation in d-dimensional Hilbert space, for which we develop the general form of the optimum POVM. Here, the conclusive measurement outcome is not to give information on the initial state itself but to give information on which unitary transformation to perform to recover the initial state at the receiving station. An optimum POVM normally leaves no information in the inconclusive event but when the POVM is used in teleportation, we show that some

teleportation information can be extracted from the inconclusive outcome. However, in this case, the maximum information the receiver recovers is limited by classical theory. If the POVM is not optimized, some *quantum* information is still left in the inconclusive event. We show how much quantum information remains in the inconclusive event by evaluating the teleportation fidelity. The overall fidelity for conclusive teleportation is less than that for the teleportation employing the von-Neumann measurement. We closely examine the reason. We also assess the minimum resources to achieve the conclusive teleportation.

II. CONCLUSIVE TELEPORTATION

Conclusive teleportation assures some faithful teleportation events using a partially-entangled quantum channel. In this section, we formulate conclusive teleportation in the d-dimensional Hilbert space $\mathcal{H}$. One of the important conditions for teleportation is that the initial quantum state is unknwon. Assume that a particle in a d-dimensional quantum state $|\phi\rangle_1$,

$$|\phi\rangle_1 = \sum_{i=1}^{d} c_i |i\rangle_1, \tag{1}$$

is teleported to a remote place via a partially entangled channel in the $d \times d$-dimensional Hilbert space, $\mathcal{H} \otimes \mathcal{H}$.

When the quantum channel is not perfect, Mor and Horodecki [5] found for a two-level quantum system that a joint POVM may assure conclusive teleportation. The main difference between a POVM and a von-Neumann orthogonal measurement is that the outcome bases and the number of outcomes may be different each other [17]. By discerning non-orthogonal states using a joint POVM, a faithful transmission of a two-level quantum state is made possible. Of course, it is not always possible to discern non-orthogonal states due to the overlap between the non-orthogonal outcome bases. For the inconclusive outcome, the teleportation becomes unfaithful. In any cases, we know when the teleportation is faithful. We formulate the optimal conclusive teleportation for a quantum state in d-dimensional Hilbert space.

A. Formalism

For quantum teleportation, we assume that the quantum channel is prepared with a *pure entangled pair* of particles 2 and 3. The quantum channel is in the state $|\psi\rangle_{23}$ by Schmidt decomposition

$$|\psi\rangle_{23} = \sum_{i=1}^{d} a_i |ii\rangle_{23} \tag{2}$$

where $\{|i\rangle\}$ is an orthonormal basis set in d-dimensional Hilbert space $\mathcal{H}$. The coefficients a_i are regarded as real values and assume that a_k is the smallest among the coefficients for the sake of simplicity. For a maximally entangled quantum channel, $a_i = d^{-1/2}$. The total quantum state for the unknown particle 1 and the entangled pair 2 and 3 is given by the direct product of the unknown state $|\phi\rangle_1$ and the quantum channel state $|\psi\rangle_{23}$,

$$|\Psi\rangle_{123} = |\phi\rangle_1 \otimes |\psi\rangle_{23}. \tag{3}$$

The sender performs a joint measurement on particles 1 and 2 so that we expand the composite system based on states for particles 1 and 2.

In the standard teleportation protocol suggested by Bennett *et al.* [1], the joint measurement is based on Bell states, which form an orthonormal basis set of the maximally entangled states $\{|\psi^m_\alpha\rangle\}$ for a spin-1/2 system. This basis set can be obtained by applying a set of local unitary operations $\{\hat{U}^\alpha\}$ on the given maximally entangled state $|\psi^m\rangle$: $|\psi^m_\alpha\rangle = \hat{U}^\alpha \otimes \mathbb{1}|\psi^m\rangle$ [18]. For a spin-1/2 system the set of unitary operators is given by $\{\mathbb{1}, \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\}$. The orthonormal basis set should satisfy the completeness:

$$\begin{aligned}\mathbb{1}_{d\times d} &= \sum_{\alpha=1}^{d^2} |\psi^m_\alpha\rangle\langle\psi^m_\alpha| \\ &= \sum_{\alpha} \hat{U}^\alpha \otimes \mathbb{1}|\psi^m\rangle\langle\psi^m|\hat{U}^{\alpha^\dagger} \otimes \mathbb{1},\end{aligned} \tag{4}$$

where $\mathbb{1}_{d\times d} = \mathbb{1} \otimes \mathbb{1}$. In the orthonormal basis of $\{|ij\rangle\}$ the completeness can be written in the matrix form:

$$\delta_{ik}\delta_{jl} = \frac{1}{d}\sum_{\alpha} U^\alpha_{ij} U^{\alpha^*}_{kl} \tag{5}$$

where $U^\alpha_{ij} \equiv \langle i|\hat{U}^\alpha|j\rangle$. Note that Eq. (5) depends only on the unitary operators. In fact the set of the unitary operators $\{\hat{U}^\alpha\}$ is the *irreducible projective representation* of a group G and Eq. (5) implies its orthogonality relation [19]. In addition the orthonormality of the entangled basis states $\delta_{\alpha\beta} = \langle\psi^m_\alpha|\psi^m_\beta\rangle$ leads to the orthogonal condition for the unitary operators:

$$\mathrm{Tr}\hat{U}^{\alpha^\dagger}\hat{U}^\beta = d\ \delta_{\alpha\beta}. \tag{6}$$

In the conclusive teleportation protocol, the basis set of entangled states is obtained by using the same local unitary operators $\{\hat{U}^\alpha\}$ on the state of the quantum channel (2):

$$|\psi_\alpha\rangle = \hat{U}^\alpha \otimes \mathbb{1}|\psi\rangle \tag{7}$$

for $\alpha = 1, 2, \cdots, d^2$. Note that the state vectors (7) are linearly independent and form a basis set. The basis states $|\psi_\alpha\rangle$ are not necessarily orthonormal. Only when the channel is maximally entangled with $a_i = d^{-1/2}$ the set of the basis states $\{|\psi_\alpha\rangle\}$ becomes $\{|\psi^m_\alpha\rangle\}$, which represents the von-Neumann orthogonal measurement.

It is possible to write the state $|\psi_\alpha\rangle$ in the orthonormal basis $|ij\rangle$:

$$|\psi_\alpha\rangle = \sum_{i,j=1}^{d} \Gamma^\alpha_{ij} |ij\rangle \tag{8}$$

where the matrix Γ^α_{ij} is calculated from Eqs. (2) and (7) as

$$\Gamma^\alpha_{ij} = U^\alpha_{ij} a_j. \tag{9}$$

Because the basis states $\{|\psi_\alpha\rangle\}$ are linearly independent, the matrix Γ^α_{ij} is non-singular and its inverse matrix is given by

$$\left(\Gamma^{-1}\right)^\alpha_{ij} = \frac{1}{d} U^{\alpha^*}_{ij} a_j^{-1}. \tag{10}$$

It is straightforward to show, from Eqs. (9) and (10) by using Eq. (5) and/or Eq. (6), that Eq. (10) is the inverse matrix for Γ^α_{ij} such that $\sum_\alpha (\Gamma^{-1})^\alpha_{ij} \Gamma^\alpha_{kl} = \delta_{ik}\delta_{jl}$ and/or $\sum_{ij} \Gamma^\alpha_{ij} (\Gamma^{-1})^\beta_{ij} = \delta_{\alpha\beta}$. With the help of $(\Gamma^{-1})^\alpha_{ij}$, the inverse relation to Eq. (8) follows

$$|ij\rangle = \sum_{\alpha=1}^{d^2} \left(\Gamma^{-1}\right)^\alpha_{ij} |\psi_\alpha\rangle. \tag{11}$$

The completeness with respect to the entangled states $\{|\psi_\alpha\rangle\}$ is nontrivial due to their being non-orthogonal for a partially entangled quantum channel. Instead, we modify the completeness for the set of orthonormal bases $\{|ij\rangle\}$ using Eq. (11) as

$$\begin{aligned} \mathbb{1}_{d\times d} &= \sum_{ij} |ij\rangle\langle ij| \\ &= \sum_{\alpha ij} \left(\Gamma^{-1}\right)^\alpha_{ij} |\psi_\alpha\rangle\langle ij|. \end{aligned} \tag{12}$$

The total state $|\Psi\rangle_{123}$ in Eq. (3) can now be written with help of the modified completeness (12) as

$$\begin{aligned} |\Psi\rangle_{123} &= \left[\sum_{\alpha ij} \left(\Gamma^{-1}\right)^\alpha_{ij} |\psi_\alpha\rangle_{12}\langle ij|\right] |\phi\rangle_1 \otimes |\psi\rangle_{23} \\ &= \frac{1}{d} \sum_\alpha |\psi_\alpha\rangle_{12} \otimes \hat{U}^{\alpha\dagger} |\phi\rangle_3. \end{aligned} \tag{13}$$

The second equality is given by the inverse matrix (10) and the orthonormality of the bases $\{|i\rangle\}$. Here it is seen that the von-Neumann orthogonal measurement with the maximally entangled bases of $\{|\psi^m_\alpha\rangle\}$ cannot exactly discern the non-orthogonal state vectors $|\psi_\alpha\rangle$ and the teleportation is no longer perfect for the partially entangled quantum channel.

Suppose that there were a complete set of measurement operators $\{\hat{A}_\alpha\}$ with d^2 outcomes, which could identify a partially entangled state $|\psi_\beta\rangle$ such that the probability $p(\alpha|\beta) = \langle\psi_\beta|\hat{A}_\alpha|\psi_\beta\rangle$ does not vanish only when $\alpha = \beta$. One could then achieve perfect teleportation by applying a unitary operation $\hat{U}_\beta$ on particle 3, accordingly. Except for the maximally entangled quantum channel, there does not exist such a set of measurement operators $\{\hat{A}_\alpha\}$. We thus adopt a set of POVM operators $\{\hat{M}_\alpha\}$ with $n > d^2$ outcomes, which are designed to satisfy the above condition of $p(\alpha|\beta) = \langle\psi_\beta|\hat{M}_\alpha|\psi_\beta\rangle \propto \delta_{\alpha\beta}$ for $\alpha \le d^2$ and to allow that $p(\alpha|\beta)$ does not have to vanish for $d^2 < \alpha \le n$. The outcomes are classified into two events. One is *conclusive* for the outcomes of $\alpha \le d^2$ where we can achieve a perfect transfer of the unknown state. The other is *inconclusive* for $d^2 < \alpha \le n$ where the measurement does not tell its outcome precisely. In the conclusive teleportation, the joint POVM determines whether the present event is conclusive. If the measurement outcome is conclusive, a corresponding unitary operation completes the teleportation.

B. Identification by POVM

A POVM is defined as a partition of unity by the non-negative operators which are in general non-orthogonal. A set of POVM operators $\{\hat{M}_\alpha\}$ with $n > d^2$ outcomes satisfy the measurement conditions of positivity and completeness. The positivity $\hat{M}_\alpha \ge 0$ ensures the positive probability for every POVM operator. The fact that the sum of probabilities is unity leads to the completeness: $\sum_{\alpha=1}^n \hat{M}_\alpha = \mathbb{1}_{d\times d}$.

The conclusive teleportation has a crucial step to identify non-orthogonal basis states $|\psi_\beta\rangle$ for $\beta = 1, 2, ..., d^2$ as in Eq. (13). For this purpose, joint POVM operators are designed such that

$$\langle\psi_\beta|\hat{M}_\alpha|\psi_\beta\rangle \propto \delta_{\alpha\beta} \quad \text{for} \quad \alpha \le d^2. \tag{14}$$

This shows that when the measurement outcome is due to any of $\hat{M}_\alpha$ for $\alpha \le d^2$, we can conclusively determine which non-orthogonal state the system is in. On the other hand, the measurement bears an inconclusive result when the outcome is of $\hat{M}_\alpha$ for $\alpha > d^2$. In the conclusive teleportation, d^2 operators are used to identify non-orthogonal states while the other operators consisting of the measurement set represent some inconclusive mixtures.

Any set of POVM operators can be decomposed into rank-one general projectors [17]. The conclusive measurement operators are represented by general projectors as

$$\hat{M}_\alpha = \lambda_\alpha |\tilde{\psi}_\alpha\rangle\langle\tilde{\psi}_\alpha| \quad \text{for} \ \alpha \le d^2 \tag{15}$$

where the real parameter $\lambda_\alpha \ge 0$ will be determined to optimize the conclusive events while keeping the probability of the inconclusive events positive. Without loss of generality we assume that each measurement outcome is equally probable with $\lambda_\alpha = \lambda$. Unless the POVM is

to discern orthogonal states, the completeness is guaranteed only by adding an inconclusive measurement operator $\hat{M}_{d^2+1}$,

$$\hat{M}_{d^2+1} = \mathbb{1}_{d\times d} - \sum_{\alpha=1}^{d^2} \hat{M}_\alpha. \tag{16}$$

For the purpose of the identification (14), the generally non-orthogonal and unnormalized states $\{|\tilde{\psi}_\alpha\rangle\}$ in Eq. (15) are constrained to satisfy the relation

$$\langle\tilde{\psi}_\alpha|\psi_\beta\rangle = \delta_{\alpha\beta}. \tag{17}$$

To find its explicit form, we expand $|\tilde{\psi}_\alpha\rangle$ in the orthogonal basis $|ij\rangle$,

$$|\tilde{\psi}_\alpha\rangle = \sum_{i,j=1}^{d} \tilde{\Gamma}^\alpha_{ij}|ij\rangle, \tag{18}$$

where $\tilde{\Gamma}^\alpha$ is given by the relation between Γ and Γ^{-1} in Eqs. (9) and (10) as

$$\tilde{\Gamma}^\alpha_{ij} = \left(\Gamma^{-1*}\right)^\alpha_{ij} = \frac{1}{d} U^\alpha_{ij} a_j^{-1}. \tag{19}$$

It is straightforward to show that the unnormalized states $|\tilde{\psi}_\alpha\rangle$ satisfy the orthogonal condition (17).

The inconclusive measurement operator $\hat{M}_{d^2+1}$ is determined to satisfy the positivity and the condition (16). To do so the sum of the conclusive measurement operators is calculated as

$$\sum_{\alpha=1}^{d^2} \hat{M}_\alpha = \frac{\lambda}{d^2} \sum_{ijkl} \frac{1}{a_j a_l} \left(\sum_\alpha U^\alpha_{ij} U^{*\alpha}_{kl} \right) |ij\rangle\langle kl| \tag{20}$$

With use of the orthogonality of unitary operators (5), the inconclusive measurement operator is then found as

$$\hat{M}_{d^2+1} = \sum_{i,j=1}^{d} \left(1 - \frac{\lambda}{a_j^2 d}\right) |ij\rangle\langle ij|. \tag{21}$$

This is positive only when $\lambda \le a_j^2 d$ for all $j = 1, 2, ..., d$. As a_k the smallest, the condition, $0 \le \lambda \le a_k^2 d$, ensures the positivity. The operator $\hat{M}_{d^2+1}$ is diagonal and a convex combination of projectors. Note that it is possible to decompose the operator $\hat{M}_{d^2+1}$ further into d^2 general projectors. We then have a new set of POVM operators, *i.e.*, $\{\hat{M}_1, \hat{M}_2, \cdots, \hat{M}_{2d^2}\}$.

C. Optimization of conclusive events

The optimization of conclusive-event probabilities is achieved by minimizing the probability of the inconclusive event,

$$p_{d^2+1} = {}_{123}\langle\Psi|\hat{M}_{d^2+1}|\Psi\rangle_{123} = 1 - \lambda, \tag{22}$$

which shows that the maximum possible value of λ will result in the optimal conclusive teleportation. From the positivity condition for all POVM operators, it is clear that λ has the maximum value, $a_k^2 d$. Note that, with the information on the channel, it is always possible to perform a POVM which optimizes faithful teleportation.

The proposed POVM enables to identify non-orthogonal states $|\psi_\alpha\rangle$ with finite probability $p_\alpha = \lambda/d^2$ for each conclusive event. When it is employed for the joint measurement, we have a nonzero probability to teleport faithfully. When the measurement outcome is inconclusive, we simply repeat the protocol till a conclusive result is obtained.

D. Necessary resources

Our protocol for the conclusive teleportation has two distinct components from standard teleportation: POVM for joint measurement and additional classical communication whether the event is conclusive or not. The additional classical communication requires single classical bit. Here, we consider what other resources are required to implement conclusive teleportation.

Neumark's theorem enables us to construct the POVM by von-Neumann orthogonal measurement in an extended Hilbert space. The extension is done by adding an ancillary particle [17]. Let d_a the dimension of the ancillary particle. In the extended Hilbert space $\mathcal{H}_s \otimes \mathcal{H}_a$, where $\mathcal{H}_s = \mathcal{H} \otimes \mathcal{H}$ is the original Hilbert space and $\mathcal{H}_a$ the ancillary Hilbert space, the von-Neumann orthogonal measurement is represented by a set of projectors $\{\hat{P}_{ij} \otimes \hat{P}_a\}$ satisfying the completeness,

$$\sum_{ija} \hat{P}_{ij} \otimes \hat{P}_a = \mathbb{1}_{d\times d} \otimes \mathbb{1}_{d_a} \tag{23}$$

where $\hat{P}_{ij} = |ij\rangle\langle ij|$ and $\hat{P}_a = |a\rangle\langle a|$ are projectors, respectively, in $\mathcal{H}_s$ and $\mathcal{H}_a$. The set of POVM operators (15) and (16) for the joint measurement are constructed by applying unitary operation $\hat{U}$ on the composite system of the original system and the ancillary system and by projecting into the original Hilbert space as

$$\mathbb{1}_{d\times d} = \hat{P}\hat{U} \sum_{ija} |ij\rangle\langle ij| \otimes |a\rangle\langle a| \hat{U}^\dagger \hat{P} \tag{24}$$

where $\hat{P}$ is a projector into $\mathcal{H}_s$ such that $\hat{P}|ij\rangle \otimes |a\rangle = |ij\rangle$. Now, the set of operators $\{\hat{P}\hat{U}|ij\rangle\langle ij| \otimes |a\rangle\langle a|\hat{U}^\dagger\hat{P}\}$ gives the set of POVM operators $\{\hat{M}_\alpha\}$ if the following d^2 matrix equations are satisfied,

$$\left(\hat{M}_\alpha\right)_{kl,mn} = \sum_{ija} \left(\hat{P}\hat{U}\right)_{kl,ija} \left(\hat{U}^\dagger\hat{P}\right)_{ija,mn} \tag{25}$$

for $\alpha \leq d^2$, where $(\hat{M}_\alpha)_{kl,mn} = \langle kl|\hat{M}_\alpha|mn\rangle$ and $(\hat{P}\hat{U})_{kl,ija} = \langle kl|\hat{P}\hat{U}|ija\rangle$. Note that we consider only d^2 matrix equations for the POVM operators used in conclusive events because others are straightforwardly obtained from the completeness condition (24). The unitary operator has $(d\times d\times d_a)^2$ independent real variables and Eq. (25) has $d^2 \times (d^2\times d^2)$ linear equations. If $d_a \geq d$, the number of real variables are sufficient to satisfy Eq. (25). Therefore, the joint POVM requires an ancillary particle in $d_a \geq d$ dimensional Hilbert space.

III. AVERAGE FIDELITY

For a maximally entangled quantum channel the proposed conclusive teleportation becomes the standard teleportation as the joint measurement performed at the sending station becomes the orthogonal measurement. In this case the joint measurement extracts no information on the unknown quantum state as the full information is transferred to the teleported state. On the other hand, for a partially entangled quantum channel, the joint orthogonal measurement may extract some information on the unknown state and only partial information is transferred to the teleported state. The average fidelity, which tells how close the teleported state is to the original state, is not larger than $\bar{\mathcal{F}}_s = \frac{2}{3}(1+|a_1 a_2|)$ for the quantum teleportation of a two-dimensional state using the standard teleportation protocol [3,4,20]. Conclusive teleportation employs the joint POVM instead of the orthogonal measurement. The measurement is different so thus the information transfer. In this section we discuss the flow of information as calculating the fidelity for conclusive teleportation.

The fidelity $\mathcal{F}$ is defined by the overlap between the original state $|\phi\rangle$ and the evolved state $\hat{\rho}$; $\mathcal{F} = \langle\phi|\hat{\rho}|\phi\rangle$. When the quantum channel is pure, a pure state is recovered at the receiving station after performing one teleportation procedure. The teleported pure state of the density operator $\hat{\rho}_\alpha$ is dependent on the measurement outcome, here, indexed α, at the sending station. After executing the teleportation protocol infinite times, the ensemble of teleported quantum system is represented by a density operator $\hat{\rho} = \sum_\alpha p_\alpha \hat{\rho}_\alpha$ where the measurement bears the outcome indexed α with the probability p_α. The fidelity can thus be given by $\mathcal{F} = \sum_\alpha p_\alpha \langle\phi|\hat{\rho}_\alpha|\phi\rangle$. In quantum teleportation, the original state $|\phi\rangle$ is unknown so that it is necessary to average the fidelity over all possible unknown states. The average fidelity is

$$\bar{\mathcal{F}} \equiv \frac{1}{V}\int d\vec{\Omega} \sum_\alpha p_\alpha(\vec{\Omega}) f_\alpha(\vec{\Omega}) \tag{26}$$

where $f_\alpha(\vec{\Omega}) = \langle\phi(\vec{\Omega})|\hat{\rho}_\alpha|\phi(\vec{\Omega})\rangle$ and an unknown pure states $|\phi(\vec{\Omega})\rangle$ is parameterized by a real vector $\vec{\Omega}$ in the parameter space of volume V [4,20,21].

In conclusive teleportation, we know that faithful teleportation is assured at the conclusive event. Even though the inconclusive result is not of use in quantum sense, the receiver can still try to recover some information on the original unknown state. To consider conclusive and inconclusive events, we decompose further the measurement operator (21) into general projectors such that the new set of POVM operators is represented by

$$\hat{M}'_\alpha = \lambda'_\alpha |\tilde{\psi}'_\alpha\rangle\langle\tilde{\psi}'_\alpha| \tag{27}$$

where

$$\lambda'_\alpha = \begin{cases} \lambda & \text{for } \alpha \leq d^2 \\ 1-\lambda/da_j^2 & \text{for } \alpha = d^2+(j-1)d+i \end{cases}$$
$$|\tilde{\psi}'_\alpha\rangle = \begin{cases} |\tilde{\psi}_\alpha\rangle & \text{for } \alpha \leq d^2 \\ |ij\rangle & \text{for } \alpha = d^2+(j-1)d+i. \end{cases}$$

The average fidelity is now written as

$$\begin{aligned}\bar{\mathcal{F}} &= \bar{\mathcal{F}}_{con} + \bar{\mathcal{F}}_{inc} \\ &= \frac{1}{V}\int d\vec{\Omega}\Big[\sum_{\alpha=1}^{d^2} + \sum_{\alpha=d^2+1}^{2d^2}\Big] p_\alpha(\vec{\Omega}) f_\alpha(\vec{\Omega})\end{aligned} \tag{28}$$

where the first (second) sum indicates the information transferred in conclusive (inconclusive) events. The average fidelity is maximized by proper unitary operator $\hat{U}_\alpha$ according to the joint measurement outcome for $\hat{M}'_\alpha$. The resulting maximal average fidelity is given by

$$\bar{\mathcal{F}} = \lambda + \frac{1-\lambda}{d+1} + \frac{1}{d(d+1)}\left(\sum_{i=1}^{d}\sqrt{da_i^2-\lambda}\right)^2. \tag{29}$$

Note that Eq. (29) is obtained using the set of the POVM operators $\{\hat{M}'_\alpha\}$. For the conclusive events the POVM operators are pre-determined. On the other hand, the inconclusive operators may vary to increase the fidelity. It was shown that the fidelity of teleportation is maximized by orthogonal joint measurement and unitary transformation [4], which is consistent with our choice of $\{\hat{M}'_\alpha\}$ for inconclusive events.

Information loss

For the comparison with the standard teleportation we restrict, in this section, our concern to two dimensional conclusive teleportation where the parameter space of unknown states is the surface of a three-dimensional sphere with its volume $V = 4\pi$. When the quantum channel is in the entangled state $|\psi\rangle_{23} = 2^{-1/2}[\sqrt{1-\cos\theta_c}|11\rangle + \sqrt{1+\cos\theta_c}|22\rangle]$, $|\tilde{\psi}'_\alpha\rangle_{12}$ in Eq. (27) are given by

$$\begin{aligned}|\tilde{\psi}'_1\rangle_{12} &= N\left[\sqrt{1+\cos\theta_c}|11\rangle + \sqrt{1-\cos\theta_c}|22\rangle\right], \\ |\tilde{\psi}'_2\rangle_{12} &= N\left[\sqrt{1+\cos\theta_c}|11\rangle - \sqrt{1-\cos\theta_c}|22\rangle\right], \\ |\tilde{\psi}'_3\rangle_{12} &= N\left[\sqrt{1+\cos\theta_c}|21\rangle + \sqrt{1-\cos\theta_c}|12\rangle\right], \\ |\tilde{\psi}'_4\rangle_{12} &= N\left[\sqrt{1+\cos\theta_c}|21\rangle - \sqrt{1-\cos\theta_c}|12\rangle\right],\end{aligned} \tag{30}$$

where $N = (2-2\cos^2\theta_c)^{-1/2}$ and $|\tilde{\psi}'_\alpha\rangle_{12} = |ij\rangle$ for $\alpha > 4$. The subscript indices 12 have been added to emphasize that the measurement is performed on particles 1 and 2. The non-orthogonality of POVM operators is implied in the relative angle θ_c between $|\tilde{\psi}'_{1(3)}\rangle$ and $|\tilde{\psi}'_{2(4)}\rangle$.

The fidelity of each conclusive event is unity with the event probability $\lambda/4$; $\bar{\mathcal{F}}_{con} = \lambda$. For the inconclusive event α, the teleported state is represented by the density operator $\hat{\rho}_\alpha = \frac{1}{2}(\mathbb{1} + \vec{m}_\alpha \cdot \vec{\sigma})$ where $\vec{m}_5 = \hat{z}$, $\vec{m}_6 = -\hat{z}$, $\vec{m}_7 = -\hat{z}$, and $\vec{m}_8 = \hat{z}$, are the four Bloch vectors. The average fidelity over the inconclusive ensemble is

$$\bar{\mathcal{F}}_{inc} = \frac{1}{2}(1-\lambda) + \frac{1}{12}\Big[\left(a_1^2 - \frac{\lambda}{2}\right)(m_5^z - m_8^z) + \left(a_2^2 - \frac{\lambda}{2}\right)(m_7^z - m_6^z)\Big]. \tag{31}$$

Because $\lambda \leq 2\,\min\{a_1^2, a_2^2\}$ due to the positivity of the measurement operators, Eq. (31) is maximized to $\bar{\mathcal{F}}_{inc} = 2(1-\lambda)/3$ by applying σ_x on the receiver's particle for the outcomes $\alpha = 7, 8$. Note that each inconclusive event has the fidelity $2/3$ on average, which is larger than the fidelity of $1/2$ in any random guess process. We have found that even the inconclusive events transfer some information of unknown states to the receiving station.

The maximum average fidelity is finally obtained for a given parameter λ as

$$\bar{\mathcal{F}} = \frac{2}{3}\left(1 + \frac{\lambda}{2}\right). \tag{32}$$

It is interesting to note that the parameter $\lambda = 2\min\{a_1^2, a_2^2\}$ optimizes not only the probabilities for conclusive events but also the average fidelity for conclusive teleportation under the condition of identification (14). The average fidelity $\bar{\mathcal{F}}_o = \frac{2}{3}(1+2\min\{a_1^2, a_2^2\})$ for optimal conclusive teleportation is clearly less than the average fidelity $\bar{\mathcal{F}}_s$ for standard teleportation. The optimal conclusive teleportation has a smaller average fidelity than the standard teleportation when the quantum channel is partially entangled.

Why does the conclusive teleportation have a smaller average fidelity? We will show that the information loss is caused by the non-orthogonality in the joint POVM. To see this we release the condition (30) for conclusive identification and we vary the relative angle $\theta_c \to \theta$. Now, the new measurement set represents also a POVM but does not necessarily give us any conclusive result. After some calculation, we find that, for a given θ, the new set of POVM operators leads to the maximal average fidelity

$$\bar{\mathcal{F}} = \frac{2}{3}\left(1 + \frac{\lambda}{2}\frac{\sqrt{1-\cos^2\theta_c}}{\sqrt{1-\cos^2\theta}}\right), \tag{33}$$

where the real parameter λ is $0 \leq \lambda \leq 1 - |\cos\theta|$. For each case of the new POVM set, the optimal fidelity $\bar{\mathcal{F}}_o$ is obtained when $\lambda = 1 - |\cos\theta|$. Fig. 1 shows the dependence of the optimal fidelity $\bar{\mathcal{F}}_o$ on the overlap, $\cos\theta$, of the POVM operators. This clearly illustrates the dependence of teleported information on the non-orthogonal nature of measurement operators. Four sets of data are plotted with respect to the degree of channel entanglement. The degree of channel entanglement is measured by the von-Neumann entropy $\mathcal{S}_a \equiv \mathrm{Tr}_a\hat{\rho}_a \log_2\hat{\rho}_a$ where $\hat{\rho}_a$ is the reduced density operator for particle a. When the overlap between the measurement bases is zero, *i.e.*, $\cos\theta = 0$, the POVM set corresponds to the orthogonal measurement. The arrows in Fig. 1 indicate the fidelities of optimized conclusive teleportation for given quantum channels. The dot-dashed line corresponds to the maximally entangled channel and gives the unit fidelity for the orthogonal measurement of $\cos\theta_c = 0$ as it should in the standard teleportation. When the non-orthogonality increases in the measurement, *i.e.*, $\cos\theta$ increases, the average fidelity decreases. We can thus say that the information of the unknown state is lost through the non-orthogonal joint measurement in teleportation.

IV. REMARKS

We have formulated the conclusive teleportation protocol utilizing the joint POVM for an unknown state in the d-dimensional Hilbert space. The general schemes are proposed on the identification of the non-orthogonal states by POVM operators and the optimization of the probabilities for the conclusive events. By the conclusive teleportation one can teleport perfectly the unknown quantum state with finite probability. It is shown that some useful information for teleportation can be extracted from an inconclusive event even when the POVM is optimized. The maximum fidelity for conclusive teleportation has been found and the minimum resources have been discussed. The fidelity for conclusive teleportation is less than the standard teleportation. We attribute the reason for the loss of information to the non-orthogonality of joint POVM operators.

ACKNOWLEDGMENTS

This work has been supported by the BK21 Grant of the Korea Ministry of Education.

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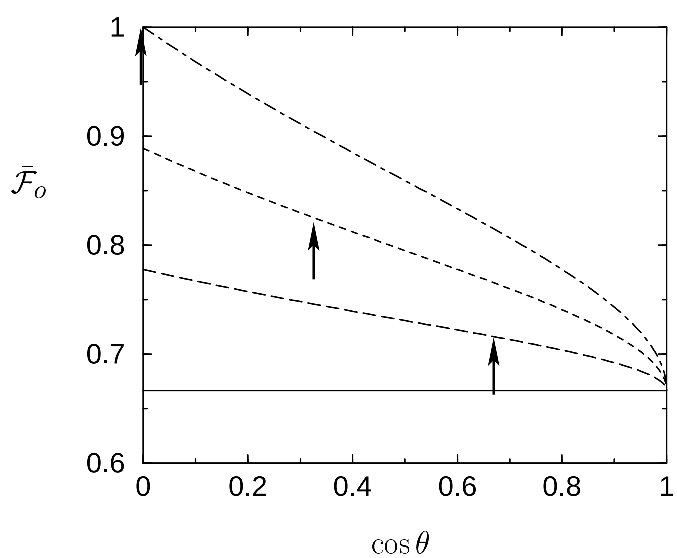

FIG. 1. The change of average optimum fidelity with regard to the relative angle of POVM operators for given entanglement of the quantum channel. The degree of entanglement for the quantum channel is measured by the von-Neumann entropy $\mathcal{S}_a = 0$ (solid line), $\mathcal{S}_a = 0.19$ (long dashed), $\mathcal{S}_a = 0.55$ (dashed), and $\mathcal{S}_a = 1$ (dot dashed). The value of $\cos\theta = 0$ corresponds to standard teleportation. The arrows indicate the optimum fidelities of conclusive teleportation for given entanglement of the quantum channel.